Floyd-Warshall

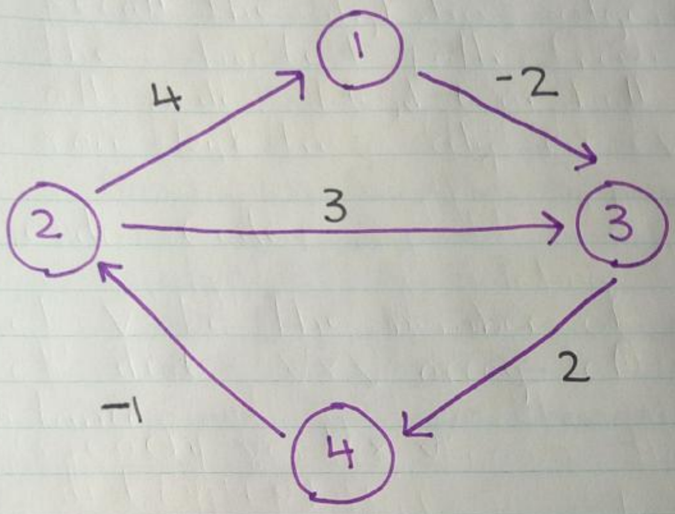
1. Definition:
   - Used to compute the shortest path between all pairs of vertices.
   - Recall the Dijkstra's algorithim calculated the shortest path from one node to all nodes. Floyd-Warshall is like if you ran Dijkstra's alg on every node.

2. Pseudo-Code:

```
Let V be the number of vertices.
Let dist = V×V array of minimum
distances.
For each vertex v:
   dist [v] [v] = 0
For each edge (u,v):
   if there is a direct path from u to v:
      dist [u] [v] = weight (u,v)
   else:
      dist[u] [v] = infinity
For k from 1 to V:
   For i from 1 to V:
      For j from 1 to V:
         if dist[i][j] > dist [i] [k] + dist [k][j]:
            dist[i] [j] = dist [i] [k] + dist [k] [j]
```

3. Example:

Consider the graph below. Use F-W to get the shortest path between all pairs of vertices.



Soln:

1. Construct the V x V array. Here, since we have 4 nodes, V=4. Furthermore, we fill all the [i][i] elements as 0 and if there is no direct path from node u to node v, [u][v] is set to infinity.

	1	2	3	4
1	0	∞	-2	∞
2	4	0	3	∞
3	∞	∞	0	2
4	∞	-1	∞	0

$= A^0$

2. Start with the node 1. Here, row and column 1 do not get affected at all.

$$A' = \begin{array}{c|c|c|c|c} & 1 & 2 & 3 & 4 \\ 1 & 0 & \infty & -2 & \infty \\ 2 & 4 & 0 & 2 & \infty \\ 3 & \infty & \infty & 0 & 2 \\ 4 & \infty & -1 & \infty & 0 \end{array}$$

Now, we do the following:

1. $\left.\begin{array}{l} \text{dist}[2][3] = 3 \\ \text{dist}[2][1] = 4 \\ \text{dist}[1][3] = -2 \end{array}\right\} 4+(-2)=2$

   Since $3>2$, we change dist[2][3] to 2.

2. $\left.\begin{array}{l} \text{dist}[2][4] = \infty \\ \text{dist}[2][1] = 4 \\ \text{dist}[1][4] = \infty \end{array}\right\} 4+\infty=\infty$

   $\infty=\infty$, so we don't change anything.

3. $\left.\begin{array}{l} \text{dist}[3][2] = \infty \\ \text{dist}[3][1] = \infty \end{array}\right\}$ We already know that nothing will change. We don't need to continue further.

4. $\left.\begin{array}{l} \text{dist}[3][4] = 2 \\ \text{dist}[3][1] = \infty \end{array}\right\}$ Nothing changes.

5. $\left.\begin{array}{l} \text{dist}[4][2] = -1 \\ \text{dist}[4][1] = \infty \end{array}\right\}$ Nothing changes.

6. dist[4][3] = ∞
   dist[4][1] = ∞ } Nothing changes

3. Move to the node 2. Now, elements in row and column 2 aren't affected.

$$A^2 = \begin{array}{c|c|c|c|c|} & 1 & 2 & 3 & 4 \\ 1 & 0 & \infty & -2 & \infty \\ 2 & 4 & 0 & 2 & \infty \\ 3 & \infty & \infty & 0 & 2 \\ 4 & 3 & -1 & 1 & 0 \end{array}$$

Now we do:

1. dist[1][3] = -2
   dist[1][2] = 4
   dist[2][3] = 2 } 4+2=6

   -2<6, so nothing changes.

2. dist[1][4] = ∞
   dist[1][2] = 4
   dist[2][4] = ∞ } 4+∞ = ∞ Nothing changes

3. dist[3][1] = ∞
   dist[3][2] = ∞
   dist[2][1] = 4 } Nothing changes

4. dist[3][4] = 2
   dist[3][2] = ∞ Nothing happens.

5. dist[4][1] = ∞
   dist[4][2] = -1
   dist[2][1] = 4 } -1+4=3
   ∞ > 3, so dist[4][1] = 3

6. dist[4][3] = ∞
   dist[4][2] = -1
   dist[2][3] = 2

   $\left.\begin{array}{l} \text{dist}[4][2] = -1 \\ \text{dist}[2][3] = 2 \end{array}\right\} -1+2=1$

   ∞ < 1, so dist[4][3] changes to 1.

4. Move to the node 3. Here, elements in row 3 and col 3 aren't affected.

$$A^3 = \begin{array}{c|c|c|c|c|} & 1 & 2 & 3 & 4 \\ 1 & 0 & \infty & -2 & 0 \\ 2 & 4 & 0 & 2 & 4 \\ 3 & \infty & \infty & 0 & 2 \\ 4 & 3 & -1 & 1 & 0 \end{array}$$

Now we do:

1. $\left.\begin{array}{l} \text{dist}[1][2] = \infty \\ \text{dist}[1][3] = -2 \\ \text{dist}[3][2] = \infty \end{array}\right\}$ Nothing changes.

2. dist[1][4] = ∞

   $\left.\begin{array}{l} \text{dist}[1][3] = -2 \\ \text{dist}[3][4] = 2 \end{array}\right\} -2+2=0$

   ∞ > 0, so dist[1][4] = 0

3. dist[2][1] = 4

   $\left.\begin{array}{l} \text{dist}[2][3] = 2 \\ \text{dist}[3][1] = \infty \end{array}\right\}$ No change.

4. dist[2][4] = ∞

   $\left.\begin{array}{l} \text{dist}[2][3] = 2 \\ \text{dist}[3][4] = 2 \end{array}\right\} 2+2=4$

   ∞ > 4, so dist[2][4] = 4

5. dist[4][1] = 3
   dist[4][3] = 1
   dist[3][1] = ∞ } No change

6. dist[4][2] = −1
   dist[4][3] = 1
   dist[3][2] = ∞ } No change

5. Move to node 4. Here, the elements in row and col 4 aren't affected.

	1	2	3	4
1	0	−1	−2	0
2	4	0	2	4
3	5	1	0	2
4	3	−1	1	0

Now we do:

1. dist[1][2] = ∞
   dist[1][4] = 0
   dist[4][2] = −1 } 0 + (−1) = −1
   ∞ > −1, so dist[1][2] = −1

2. dist[1][3] = −2
   dist[1][4] = 0
   dist[4][3] = 1 } 0+1=1
   −2 < 1, so nothing happens.

3. dist[2][1] = 4
   dist[2][4] = 4
   dist[4][1] = 3 } 4+3=7
   4<7, so nothing changes.

4. dist[2][3] = 2
   dist[2][4] = 4
   dist[4][3] = 1 } 4+1=5
   2<5, so nothing happens.

5. dist[3][1] = ∞
dist[3][4] = 2
dist[4][1] = 3
$\left.\begin{matrix} \text{dist}[3][4] = 2 \\ \text{dist}[4][1] = 3 \end{matrix}\right\} 2+3=5$
∞ > 5, so dist[3][1] = 5

6. dist[3][2] = ∞
$\left.\begin{matrix} \text{dist}[3][4] = 2 \\ \text{dist}[4][2] = -1 \end{matrix}\right\} 2+(-1)=1$
∞ > 1, so dist[3][2] = 1

The final matrix is:

	1	2	3	4
1	0	-1	-2	0
2	4	0	2	4
3	5	1	0	2
4	3	-1	1	0

4. Complexity:

- Takes $O(v^3)$, where v is the number of vertices.